# Nonlinear ion-acoustic waves in a collisional plasma

S. V. Vladimirov* and M. Y. Yu
*Institut für Theoretische Physik I, Ruhr-Universität Bochum, D-44780 Bochum, Germany*
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It is shown that finite-amplitude ion-acoustic waves in a collision-dominated plasma are described by a Korteweg–de Vries–Burger equation, which has the same form, but with very different scaling and parameter dependence from that describing ion-acoustic waves in a weakly collisional plasma.

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## I. INTRODUCTION

Processes in low-temperature plasmas which are collision dominated have been of recent interest because of their relevance in many modern technological applications of plasmas. Because of the rapid momentun and energy transfer between the particles, highly collisonal plasmas have the common properties that dissipative effects are always present, and the electrons and ions can have similar temperatures. One of the most important problems in collisional plasmas is that of wave propagation [1–7], since the stability and nonlinear behavior of the waves can crucially affect the property of the plasma.

A commonly occurring mode in both collisionless and collisional plasmas is the ion-acoustic wave. In a collisionless and nonisothermal plasma where the electron temperature is much larger than the ion temperature ($T_e \gg T_i$), these waves are driven by the electron pressure and ion inertia, the coupling between the species being achieved by the electrostatic forces. Although the dispersion relation remains similar to that of the collisionless case, the physics of the ion-acoustic waves in a highly collisional nonisothermal plasma is more complicated, since both electrostatic and collisional effects enter into play. For example, collisions between the unlike particles can also couple the dynamics of the ions and the electrons. Thus, the collisional ion-acoustic waves can involve both plasma and neutral-fluid properties. Furthermore, collision-driven resistive and dissipative instabilities can occur if external free-energy sources, such as external currents, density and velocity inhomogeneities, etc., are present [1–5], and the waves can become nonlinear and/or turbulent.

Although there exists a large number [1,5–9] of studies on nonlinear ion-acoustic waves in collisionless and weakly collisional plasmas, there seems to be no comprehensive investigation of such waves in a strongly collisional plasma. In fact, a consistent (i.e., including dispersion as well as dissipation) study of the linear ion-acoustic waves in such plasmas does not seem to exist in the literature. In this paper, we consider the nonlinear propagation of ion-acoustic waves in a collision-dominated plasma, taking into account the variations of the particle densities, fluid velocities, as well as temperatures in the wave field. It is found that the propagation is governed by the Korteweg–de Vries–Burger (KdVB) equation [6,7], similar to that for the weakly collisional plasmas. However, here the scaling and therefore the physics are completely different from the collisional case, Also, in contrast to the latter, where the nonlinearity originates mainly from ion convection and electron pressure, here it is dominated by the thermal forces and inter-particle heat transfer. The KdVB equation, which in the present case is not reducible to the KdV equation because of the scaling, admits shocklike solutions which differ from the usual shock waves by having a decaying oscillating tail in the downstream region [1,5–7].

## II. BASIC EQUATIONS

Unlike the case for hot, nearly collisionless plasmas, where the electrons are in thermal equilibrium and are governed by the Boltzmann distribution, here the full dynamics of both the ions and electrons must be considered. Accordingly, we start with the equations for the fluid velocities $\mathbf{v}_e$ and $\mathbf{v}_i$ of the electrons and ions [10]:

$$m_e n_e(\partial_t + \mathbf{v}_e \cdot \boldsymbol{\nabla}) v_{e;j} = -\nabla_j n_e T_e - \nabla_l \pi_{lj}^{(e)} - e n_e E_j + R_j, \qquad (1)$$

and

$$m_i n_i(\partial_t + \mathbf{v}_i \cdot \boldsymbol{\nabla}) v_{i;j} = -\nabla_j n_i T_i - \nabla_l \pi_{lj}^{(i)} + e n_i E_j - R_j, \qquad (2)$$

where the subscripts or superscripts $e$ and $i$ denote electron and ion quantities, respectively, and $l$ and $j$ are dummy spatial-direction indices. Furthermore, $\pm e$, $m$, $n$, $\mathbf{v}$, and $T$ are the charges, masses, densities, fluid velocities, and temperatures of the species, and $\mathbf{E}$ is the electric field. Equations (1) and (2) are completed by the continuity equations

$$\partial_t n_{e,i} + \boldsymbol{\nabla} \cdot (n_{e,i} \mathbf{v}_{e,i}) = 0, \qquad (3)$$

and the energy balance equations

$$\tfrac{3}{2} n_{e,i}(\partial_t + \mathbf{v}_{e,i} \cdot \boldsymbol{\nabla}) T_{e,i} + n_{e,i} T_{e,i} \boldsymbol{\nabla} \cdot \mathbf{v}_{e,i} = -\boldsymbol{\nabla} \cdot \mathbf{q}^{(e,i)} - \pi_{lj}^{(e,i)} \boldsymbol{\nabla} v_{e,i;l} + Q_{e,i}. \qquad (4)$$

 

In the above equations, the terms $\nabla nT$ represent the pressure forces of the electron and ion gases, and the stress tensors $\pi_{lj}^{(e,i)}$ are given by

$$\pi_{lj}^{(e)} = -0.73\frac{n_e T_e}{\nu_e} w_{lj}^{(e)}, \quad \pi_{lj}^{(i)} = -0.96\frac{n_i T_i}{\nu_i} w_{lj}^{(i)}, \quad (5)$$

with the rate of strain tensors $w_{lj}^{(e,i)}$ given by

$$w_{lj}^{(e,i)} = \nabla_j v_{e,i;l} + \nabla_l v_{e,i;j} - \tfrac{2}{3}\delta_{lj}\boldsymbol{\nabla}\cdot\mathbf{v}_{e,i}. \quad (6)$$

Furthermore, the friction force $\mathbf{R}$ between the electrons and ions is

$$\mathbf{R} = \mathbf{R}_{\mathbf{u}} + \mathbf{R}_T, \quad (7)$$

where $\mathbf{R}_{\mathbf{u}}$ is associated with the force of relative friction (for $\omega \ll \nu_e$)

$$\mathbf{R}_{\mathbf{u}} = -0.51 n_e m_e \nu_e \mathbf{u}, \quad (8)$$

which depends only on the relative velocity $\mathbf{u} = \mathbf{v}_e - \mathbf{v}_e$ between the electrons and ions. Note that here the effective collision frequency is $\nu_{\text{eff}} \simeq \nu_e$. Let us also stress that in the opposite limit, namely $\omega \gg \nu_e$, one can obtain the mathematically similar relation $\mathbf{R}_{\mathbf{u}} \simeq -n_e m_e \nu_e \mathbf{u}$. However, in this case serious questions on the validity of the hydrodynamic description arise [10].

The thermal-gradient frictional force $\mathbf{R}_T$ appearing in (7) is given by

$$\mathbf{R}_T = -0.71 n_e \boldsymbol{\nabla} T_e. \quad (9)$$

Furthermore, the heat fluxes $\mathbf{q}^{(e,i)}$ are

$$\mathbf{q}^{(e)} = \mathbf{q}_{\mathbf{u}}^{(e)} + \mathbf{q}_T^{(e)} = 0.71 n_e T_e \mathbf{u} - 3.16\frac{n_e T_e}{m_e \nu_e}\boldsymbol{\nabla} T_e,$$
$$\mathbf{q}^{(i)} = -3.9\frac{n_i T_i}{m_i \nu_i}\boldsymbol{\nabla} T_i. \quad (10)$$

Finally, the heating powers $Q_{e,i}$ are

$$Q_e = -\mathbf{R}\cdot\mathbf{u} - Q_i, \quad Q_i = 3\frac{m_e}{m_i} n_e \nu_e (T_e - T_i). \quad (11)$$

In the following, we shall solve the above equations by expanding in powers of the electric field $\mathbf{E}$.

## III. LINEAR THEORY

We shall first present the linear theory, which does not seem to have appeared before in its complete form, and will be needed in our nonlinear investigation later. Assuming the Fourier-mode form $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$, we obtain straightforwardly the first-order perturbations

$$\mathbf{v}_e^{(1)} = -\frac{e\mathbf{E}_k}{m_e \kappa \omega_e} \approx \frac{ie\mathbf{E}_k}{m_i \omega}\frac{10}{5 + 2\times 0.71}, \quad \mathbf{v}_i^{(1)} = -\frac{e\mathbf{E}_k}{m_i \kappa \omega_i} \approx \mathbf{v}_e^{(1)}, \quad (12)$$

$$T_e^{(1)} = iT_0\frac{e\mathbf{k}\cdot\mathbf{E}_k}{m_e \kappa \Omega_e \Delta}\left(\frac{1.71}{\omega_e} + \frac{m_e}{m_i}\frac{1.71 - \Delta_e}{\omega_i}\right) \approx \frac{i}{5}\frac{e\mathbf{k}\cdot\mathbf{E}_k}{k^2}\frac{10}{5 + 2\times 0.71},$$
$$T_i^{(1)} = -iT_0\frac{e\mathbf{k}\cdot\mathbf{E}_k}{m_e \kappa \Omega_i \Delta}\left(\frac{1.71(1-\Delta_e)}{\omega_e} + \frac{m_e}{m_i}\frac{1.71 - 0.71\Delta_i}{\omega_i}\right) \approx T_e^{(1)}, \quad (13)$$

where $T_0$ is the equilibrium temperature of the plasma. In the above, we have retained certain intermediate steps in order to identify their origin. We have also defined

$$\omega_e = -i\omega + i\frac{k^2 v_{Te}^2}{\omega}\left(1 - 1.71\frac{i\omega\Delta_e}{\Omega_e \Delta}\right) + \frac{4}{3}0.73\frac{k^2 v_{Te}^2}{\nu_e}$$
$$\approx i\frac{k^2 v_{Te}^2}{\omega}\frac{5 + 2\times 0.71}{3}, \quad (14)$$

$$\omega_i = -i\omega + i\frac{k^2 v_{Ti}^2}{\omega}\left(1 - \frac{i\omega\Delta_e}{\Omega_e\Delta} + 0.71\frac{i\omega\Delta_e}{\Omega_e\Delta}\right)$$
$$+\frac{4}{3}0.96\frac{k^2 v_{Te}^2}{\nu_e}$$
$$\approx -i\omega\left(1 - \frac{k^2 v_{Ti}^2}{\omega^2}\frac{5 - 2\times 0.71}{3}\right), \quad (15)$$

$$\Omega_e = -\frac{3}{2}i\omega + 3.16\frac{k^2 v_{Te}^2}{\nu_e} + 3\frac{m_e}{m_i}\nu_e \approx 3\frac{m_e}{m_i}\nu_e,$$

$$\Omega_i = -\frac{3}{2}i\omega + 3.9\frac{k^2 v_{Ti}^2}{\nu_i} + 3\frac{m_e}{m_i}\nu_e \approx 3\frac{m_e}{m_i}\nu_e \approx \Omega_e, \quad (16)$$

$$\Delta_{e,i} = 1 + 3\frac{m_e}{m_i}\frac{\nu_e}{\Omega_{i,e}} \approx 2, \quad (17)$$

$$\Delta = 1 - \left(3\frac{m_e}{m_i}\nu_e\right)^2\frac{1}{\Omega_e\Omega_i} \approx -\frac{m_i}{m_e}\frac{i\omega}{\nu_e}, \quad (18)$$

and

$$\kappa = 1 + \left(0.51\nu_e + 1.71\frac{k^2 v_{Te}^2}{\Omega_e}\frac{1.71 - 0.71\Delta_i}{\Delta}\right)$$
$$\times\left(\frac{1}{\omega_e} + \frac{m_e}{m_i}\frac{1}{\omega_i}\right) \approx 1, \quad (19)$$

where $v_{Te,i} = \sqrt{T_0/m_{e,i}}$ is the electron (ion) thermal velocity. Thus, we have expressed all the relevant quantities in terms of the electric field $\mathbf{E}_k$.

The corresponding linear dispersion relation is easily found to be

$$\omega_i + \frac{m_e}{m_i}\omega_e = 0. \tag{20}$$

From Eq. (20) we obtain

$$\omega \approx kv_s - iA\frac{k^2 v_{Te}^2}{\nu_e} - Bkv_s\frac{k^2 v_{Te}^2}{\nu_e^2}, \tag{21}$$

where

$$5A = \frac{3.16}{3} + \left(\frac{3.9}{3} + \frac{9.6}{3}\right)\frac{\nu_e m_e}{\nu_i m_i}, \tag{22}$$

and

$$5B = \frac{3m_i}{10m_e}\left(\frac{3.16}{3} + \frac{3.9}{3}\frac{\nu_e m_e}{\nu_i m_i}\right)^2. \tag{23}$$

Equation (21) gives the frequency of the ion-acoustic waves in a collision-dominated plasma. We see that the waves exhibit collision-driven damping and dispersion, and that the contributions of the stress tensor as well as the temperature perturbations, usually neglected, are significant.

## IV. NONLINEAR TERMS

We can conveniently express the second-order temperature perturbations as

$$T_e^{(2)} = \tilde{T}_e^{(2)} - \frac{1}{\Omega_e\Delta}[1.71T_0(\nabla\cdot\mathbf{v}_e^{(2)}) - 0.71T_0(\nabla\cdot\mathbf{v}_i^{(2)})] - 3\frac{m_e}{m_i}\frac{\nu_e T_0(\nabla\cdot\mathbf{v}_i^{(2)})}{\Omega_e\Omega_i\Delta}, \tag{24}$$

$$T_i^{(2)} = \tilde{T}_i^{(2)} - \frac{T_0(\nabla\cdot\mathbf{v}_i^{(2)})}{\Omega_i\Delta} - 3\frac{m_e}{m_i}\frac{\nu_e}{\Omega_e\Omega_i\Delta}[1.71T_0(\nabla\cdot\mathbf{v}_e^{(2)}) - 0.71T_0(\nabla\cdot\mathbf{v}_i^{(2)})], \tag{25}$$

where we have defined

$$\tilde{T}_{e,i}^{(2)} = \frac{1}{\Omega_{e,i}\Delta}\left(a_{e,i} + 3\frac{m_e}{m_i}\frac{\nu_e}{\Omega_{i,e}}a_{i,e}\right), \tag{26}$$

$$a_{e,i} = -\frac{3}{2}(\mathbf{v}_{e,i}^{(1)}\cdot\nabla)T_{e,i}^{(1)} - T_{e,i}^{(1)}(\nabla\cdot\mathbf{v}_{e,i}^{(1)}). \tag{27}$$

Then the second-order relative velocity can be conveniently expressed as

$$\mathbf{u}^{(2)} = \mathbf{v}_e^{(2)} - \mathbf{v}_i^{(2)} = -\frac{2m_i T_0}{\kappa\omega_i m_e^2 n_0}\left[\frac{n_e^{(1)}}{n_0}\nabla n_e^{(1)} - \frac{T_e^{(1)}}{T_0}\nabla n_e^{(1)} - \frac{1}{i\omega}\nabla(n_e^{(1)}\nabla\cdot\mathbf{v}_e^{(1)})\right] + \frac{1}{\kappa\omega_i}\left[(\mathbf{v}_i^{(1)}\cdot\nabla)\mathbf{v}_i^{(1)} + \frac{1}{m_i}\nabla(\tilde{T}_e^{(2)} + \tilde{T}_i^{(2)})\right], \tag{28}$$

which we note contains mainly contributions from thermal forces and heat fluxes.

Using Eqs. (3), (12), and (13) we obtain for the Fourier component [which is proportional to $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$] of $\mathbf{u}^{(2)}$

$$\mathbf{k}\cdot\mathbf{u}_k^{(2)} \simeq -\left(\frac{10}{5+2\times 0.71}\right)^2\int\frac{ie^2k^2}{m_i^2\kappa\omega_i\omega_1\omega_2 k_1k_2}E_{k_1}E_{k_2}\left[\frac{10}{9}\frac{T_e k_1^2k_2^2}{m_i\omega_1\omega_2} + \frac{1}{2}\mathbf{k}_1\cdot\mathbf{k}_2 + \frac{5}{3}\frac{T_e\mathbf{k}_1\cdot\mathbf{k}_2}{m_i\omega}\left(\frac{k_1^2}{\omega_1} + \frac{k_2^2}{\omega_2}\right)\right]d\Omega\, d\mathbf{K}, \tag{29}$$

where the subscripts 1 and 2 are dummy wave-number indices, and $d\Omega\, d\mathbf{K}$ stands for $\delta(\omega-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, d\omega_1 d\omega_2 d\mathbf{k}_1 d\mathbf{k}_2$. The Fourier component of the longitudinal electric field is $E_k = \mathbf{k}\cdot\mathbf{E}_k/k$, where $\mathbf{E}_k = \int E(\mathbf{x},t)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{x})d\mathbf{x}\, dt$.

For the one-dimensional case, we have from (29)

$$u_k^{(2)} \simeq -\left(\frac{10}{5+2\times 0.71}\right)^2 \times\int\frac{4ie^2k}{3m_i^2\omega_i\omega_1\omega_2 k_1k_2}E_{k_1}E_{k_2}d\Omega\, dK. \tag{30}$$

In obtaining (30) we have used the dispersion relation (20). Thus, Eq. (30) is valid only for waves with sound-like dispersion, as given by (21). Let us also note that terms due to the quasilinear flux $n_e^{(1)}\mathbf{v}_e^{(1)} - n_i^{(1)}\mathbf{v}_i^{(1)}$ in the expression for the second-order current density are negligibly small comparing with $\mathbf{u}^{(2)}$ given by (28).

Thus, for the (one-dimensional) electric field of the ion-acoustic waves we get the following second-order equation:

$$\varepsilon_k E_k - \left(\frac{10}{5+2\times 0.71}\right)^3\frac{2ie}{5m_i r_{De}^2 v_s^2}\times\int\frac{E_{k_1}E_{k_2}}{kk_1k_2}d\Omega\, dK = 0. \tag{31}$$

The linear dielectric permittivity $\varepsilon_k$ is given by

$$\varepsilon_k = 1 + \frac{i\omega_{pe}^2}{\kappa\omega\omega_e} + \frac{i\omega_{pe}^2}{\kappa\omega\omega_e} \simeq \frac{i\omega_{pe}^2}{\kappa\omega_e\omega_i}\left(\omega_i + \frac{m_e}{m_i}\omega_e\right), \quad (32)$$

where $\omega_{pe(i)} = (4\pi n_0^2/m_{e(i)})^{1/2}$ is the electron (ion) plasma frequency, and $r_{De} = v_{Te}/\omega_{pe}$ is the electron Debye length. From (31) one can easily obtain

$$\omega - kv_s + iA\frac{k^2 v_{Te}^2}{\nu_e} + Bkv_s\frac{k^2 v_{Te}^2}{\nu_e^2}$$

$$+\frac{40/3}{5+2\times 0.71}\frac{iek^2}{m_i v_s}\int \frac{E_{k_1}E_{k_2}}{k_1 k_2}d\Omega\, dK = 0. \quad (33)$$

It is convenient to define the electrostatic potential $\varphi$ by $E_k = -ik\varphi_k$. Inverse Fourier transforming, one obtains

$$(\partial_t + \partial_x - \partial_{xx} + 5\partial_{xxx})\phi + \partial_x(\phi)^2 = 0, \quad (34)$$

where $t$, $x$, and $\phi$ have been normalized by $3.16m_e/50m_i\nu_e$, $3.16m_e v_s/50m_i\nu_e$, and $3(5+2\times 0.71)m_i v_s^2/40e$, respectively. We also have assumed for convenience that $\nu_e m_e \ll \nu_i m_i$, so that, in for example (22), $5A \approx 3.16/3$.

Equation (34), which does not contain any dimensionless parameters, is the KdVB equation. It has been intensively studied in the literature, and has the well-known quasistationary solution depicting a shock wave with a spatially decaying and oscillating tail [5–7].

## V. DISCUSSION

Unlike the KdVB equation for a weakly collisional plasma, Eq. (34) is not reducible to a simple KdV equation in any limit. This is because here the scaling (i.e., the normalization parameters) is fixed in the present problem, as is evident from the absence of free parameters which can be rescaled in order that the dissipation term could be neglected as a result. Thus, the corresponding solutions, namely, shocklike structures with oscillating downstream tails, usually attributed to weakly collisional plasmas [1,5–7], occur in a strongly collisional plasma as a rule. It also follows that ion-acoustic solitons of the KdV type cannot appear in such plasmas. Physically, this result is expected, since when collisional effects dominate, dissipation is inevitable. The fact that thermal forces and interparticle heat transfer dominate the nonlinear mechanism is also in some sense expected, since dissipation, similar to dispersion, is particularly sensitive to the large gradients associated with the shock wave.

In this paper, we have not included effects such as external currents, background inhomogeneities, impurities, ionization, and recombination, etc., which may be of importance in a low-temperature plasma. These effects can lead to phenomena such as linear and nonlinear instabilities which may affect the formation of the stationary states. Furthermore, in the region near the container wall, boundary effects are expected to modify the properties of the nonlinear sound waves. For example, linear and nonlinear *surface* acoustic waves [11,12], which are important in many applications, can appear. This and related effects are still being investigated.

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* Permanent address: Theory Department, General Physics Institute, 117942 Moscow, Russia.

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